Spring 2017 MATH5012

Exercise 3

(1) For $a, b > 0$, set

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & 0 < x \leq 1 \\ 0, & x = 0. \end{cases}$$

Show that f is in $BV[0,1]$ iff $a > b$.

(2) A function is called Lipschitz continuous on an interval I if $\exists M > 0$ such that

$$|f(x) - f(y)| \leq M\,|x - y|\,.$$

(a) Show that every Lipschitz continuous function is absolutely continuous on I.

(b) Show that there are always absolutely continuous functions which are not Lipschitz continuous.

(3) Assume that $1 < p < \infty$, f is absolutely continuous on $[a,b]$, $f' \in L^p$, and $\alpha = 1/q$, where q is the exponent conjugate to p. Prove that $f \in \mathrm{Lip}\alpha$.

(4) Show that the product of two absolutely continuous functions on $[a,b]$ is absolutely continuous. Use this to derive a theorem about integration by parts.

(5) Suppose $E \subset [a,b]$, $m(E) = 0$. Construct an absolutely continuous monotonic function f on $[a,b]$ so that $f'(x) = \infty$ at every $x \in E$.
Hint: $E \subset \bigcap V_n$, V_n open, $m(V_n) \leq 2^{-n}$. Consider the sum of the characteristic functions of these sets.

(6) Let f be in $AC[a,b]$. Show that the total variation for f of f is also in $AC[a,b]$. Moreover,

$$T_f(b) = \int_a^b |f'(t)|\;dt.$$

(7) Let X and Y be topological spaces having countable bases.

(a) Show that $X \times Y$ has a countable base. (In product topology on $X \times Y$, a set G is open if $\forall (x,y) \in G$, $\exists G_1$ open in X, G_2 open in Y such that $(x,y) \in G_1 \times G_2 \subset G$.)

(b) Let μ and ν be Borel measures on X and Y respectively. Show that $\mu \times \nu$ is a Borel measure.

(8) Let μ be the product measure $\mathcal{L}^1 \times \cdots \times \mathcal{L}^1$ on $\mathbb{R}^n$. Show that μ is equal to $\mathcal{L}^n$.

(9) Fix $a_1 = 0 < a_2 < a_3 < \cdots < a_n \uparrow 1$ and let g_n be a continuous function, $\mathrm{spt} g_n \subset (a_n, a_{n+1})$, $n \geq 1$, $\displaystyle\int g_n = 1$. Let

$$f(x,y) = \sum_{n=1}^{\infty} (g_n(x) - g_{n+1}(x)) g_n(y).$$

Verify that

$$\int \left(\int f \, dx \right) dy = 0, \text{ but}$$
$$\int \left(\int f \, dy \right) dx = 1,$$

and f is $\mathcal{L}^2$-measurable. Explain why Fubini's theorem cannot apply.

(10) Let μ and ν be outer measures defined on X and Y respectively and let f be μ-measurable and g ν-measurable with values in $(-\infty, \infty]$. Is it true that $(x,y) \mapsto f(x) + g(y)$ measurable in $\mu \times \nu$? How about the map $(x,y) \mapsto f(x)g(y)$?

(11) (a) Suppose that f is a real-valued function in $\mathbb{R}^2$ such that each section f_x is Borel measurable and each section f^y is continuous. Prove that f is Borel measurable in $\mathbb{R}^2$. There is a hint given in [R].

(b) Suppose that g is a real-valued function in $\mathbb{R}^n$ which is continuous in each of the n-variables separately. Prove that g is Borel.

(12) Suppose that f is real-valued in $\mathbb{R}^2$, f_x is Lebesgue measurable for each x, and f^y is continuous for each y. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, and put $h(y) = f(g(y), y)$. Prove that h is Lebesgue measurable on $\mathbb{R}$. Hint: Use Problem 5.

(13) Give an example of two measurable sets A and B in $\mathbb{R}^2$ but $A + B$ is not measurable.

Suggestion: For the two-dimensional case, take $A = \{0\} \times [0, 1]$ and $B = \mathcal{N} \times \{0\}$ where $\mathcal{N}$ is a non-measurable set in $\mathbb{R}$.